

# Total Curvatures of Model Surfaces Control Topology of Complete Open Manifolds with Radial Curvature Bounded Below. III<sup>\*†</sup>

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*Dedicated to Professor K. Shiohama on the occasion of his seventieth birthday.*

## Abstract

This article is the third in a series of our investigation on a complete non-compact connected Riemannian manifold  $M$ . In the first series [KT1], we showed that all Busemann functions on an  $M$  which is not less curved than a von Mangoldt surface of revolution  $\widetilde{M}$  are exhaustions, if the total curvature of  $\widetilde{M}$  is greater than  $\pi$ . A von Mangoldt surface of revolution is, by definition, a complete surface of revolution homeomorphic to  $\mathbb{R}^2$  whose Gaussian curvature is non-increasing along each meridian. Our purpose of this series is to generalize the main theorem in [KT1] to an  $M$  which is not less curved than a more general surface of revolution.

## 1 Introduction

The Gauss–Bonnet theorem says that the total curvature  $c(S)$  of a **compact** Riemannian 2-dimensional manifold  $S$  is a topological invariant, i.e.,

$$c(S) = 2\pi\chi(S).$$

Here  $\chi(S)$  denotes the Euler characteristic of  $S$ .

In 1935, Cohn-Vossen generalized the Gauss–Bonnet theorem for complete **non-compact** Riemannian 2-dimensional manifolds as follows:

**Theorem 1.1** ([CV1, Satz 6]) *If a connected, complete non-compact, finitely connected Riemannian 2-manifold  $M$  admits a total curvature  $c(M)$ , then,*

$$c(M) \leq 2\pi\chi(M)$$

*holds. Here  $\chi(M)$  denotes the Euler characteristic of  $M$ .*

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Notice the total curvature  $c(M)$  is not a topological invariant anymore. But  $2\pi\chi(M) - c(M)$  is a geometric invariant depending only on the ends of  $M$ , which is a consequence from the isoperimetric inequalities (see [SST, Theorem 5.2.1]).

In 1984, Shiohama proved the next result peculiar to geometry of total curvature on surfaces :

**Theorem 1.2** ([S, Main Theorem]) *Let  $M$  be a connected, complete non-compact, finitely connected and oriented Riemannian 2-manifold with one end. If the total curvature  $c(M)$  satisfies*

$$c(M) > (2\chi(M) - 1)\pi,$$

*then all Busemann functions on  $M$  are exhaustions. In particular, if the total curvature of  $M$  is greater than  $\pi$ , then  $M$  is homeomorphic to  $\mathbb{R}^2$  and also all Busemann functions are exhaustions.*

Here the *Busemann function*  $F_\gamma : M \rightarrow \mathbb{R}$  of a ray  $\gamma$  in a complete non-compact Riemannian (any dimensional) manifold  $M$  is, by definition,

$$F_\gamma(x) := \lim_{t \rightarrow \infty} \{t - d(x, \gamma(t))\},$$

and a function  $\varphi : M \rightarrow \mathbb{R}$  is called an exhaustion, if  $\varphi^{-1}(-\infty, a]$  is compact for all  $a \in \mathbb{R}$ .

Theorem 1.2 was generalized to higher-dimensional manifolds in [KT1]. Roughly speaking, it was proved in [KT1] that all Busemann functions on a complete non-compact connected Riemannian manifold not less curved than a von Mangoldt surface of revolution  $\widetilde{M}$  are exhaustions, if the total curvature of  $\widetilde{M}$  is greater than  $\pi$  (The theorem will be later stated in full detail as Theorem 1.4 in this article).

A von Mangoldt surface of revolution is, by definition, a complete surface of revolution homeomorphic to  $\mathbb{R}^2$  whose Gaussian curvature is non-increasing along each meridian. The monotonicity of the Gaussian curvature of a von Mangoldt surface of revolution looks restrictive, but very familiar surfaces such as a paraboloid or a 2-sheeted hyperboloid are von Mangoldt surfaces of revolution.

Although Cohn-Vossen restricted himself to 2-dimensional manifolds, he has developed fundamental techniques, such as drawing a circle or a geodesic polygon, and joining two points by a minimal geodesic segment, to investigate the structures of complete Riemannian 2-dimensional manifolds. We, Riemannian geometers, should be awed by the fact that such techniques are ever now not only useful, but also powerful for investigating the topology of **any dimensional** complete Riemannian manifolds.

Furthermore, as pointed out in the preface of [SST], *it took more than thirty years to obtain higher-dimensional extensions of Cohn-Vossen's results* for complete non-compact Riemannian 2-dimensional manifolds. They are the splitting theorem by Toponogov [To], the structure theorem with positive sectional curvature by Gromoll and Meyer [GM], and the soul theorem with non-negative sectional curvature by Cheeger and Gromoll [CG].

Hence, it requires many years and is also very difficult to generalize some results peculiar to geometry of surfaces to any dimensional complete Riemannian manifolds. In fact, one may find such results in [SST], which have not been generalized in higher dimensions yet.

Our purpose of this article is to generalize the main theorem in [KT1] to a complete non-compact connected Riemannian manifold not less curved than a more general surface of revolution. To state this precisely, we will begin on the definition of a non-compact model surface of revolution.

Let  $\widetilde{M}$  denote a complete 2-dimensional Riemannian manifold homeomorphic to  $\mathbb{R}^2$  with a base point  $\tilde{p} \in \widetilde{M}$ . Then, we call the pair  $(\widetilde{M}, \tilde{p})$  a *non-compact model surface of revolution* if its Riemannian metric  $d\tilde{s}^2$  is expressed in terms of geodesic polar coordinates around  $\tilde{p}$  as

$$d\tilde{s}^2 = dt^2 + f(t)^2 d\theta^2, \quad (t, \theta) \in (0, \infty) \times \mathbb{S}_{\tilde{p}}^1. \quad (1.1)$$

Here  $f : (0, \infty) \rightarrow \mathbb{R}$  is a positive smooth function which is extensible to a smooth odd function around 0, and  $\mathbb{S}_{\tilde{p}}^1 := \{v \in T_{\tilde{p}}\widetilde{M} \mid \|v\| = 1\}$ . The function  $G \circ \tilde{\gamma} : [0, \infty) \rightarrow \mathbb{R}$  is called the *radial curvature function* of  $(\widetilde{M}, \tilde{p})$ , where we denote by  $G$  the Gaussian curvature of  $\widetilde{M}$ , and by  $\tilde{\gamma}$  any meridian emanating from  $\tilde{p} = \tilde{\gamma}(0)$ . Remark that  $f$  satisfies the differential equation

$$f''(t) + G(\tilde{\gamma}(t))f(t) = 0$$

with initial conditions  $f(0) = 0$  and  $f'(0) = 1$ . For each constant number  $\delta > 0$ , a sector  $\widetilde{V}(\delta) \subset \widetilde{M}$  is defined by

$$\widetilde{V}(\delta) := \left\{ \tilde{x} \in \widetilde{M} \mid 0 < \theta(\tilde{x}) < \delta \right\}.$$

Notice that the  $n$ -dimensional model surfaces of revolution are defined similarly, and they are completely classified in [KK].

The total curvature  $c(\widetilde{M})$  of  $(\widetilde{M}, \tilde{p})$  is formally defined as the improper integral, i.e.,

$$c(\widetilde{M}) := \int_{\widetilde{M}} G_+ \circ t \, d\widetilde{M} + \int_{\widetilde{M}} G_- \circ t \, d\widetilde{M}$$

if

$$\int_{\widetilde{M}} G_+ \circ t \, d\widetilde{M} < \infty, \quad \text{or} \quad \int_{\widetilde{M}} G_- \circ t \, d\widetilde{M} > -\infty.$$

Here we set

$$G_+(t) := \max\{G(\tilde{\gamma}(t)), 0\} = \frac{G + |G|}{2}$$

and

$$G_-(t) := \min\{G(\tilde{\gamma}(t)), 0\} = \frac{G - |G|}{2}.$$

Notice that  $G = G_+ \circ t + G_- \circ t$ . If  $c(\widetilde{M})$  exists,  $c(\widetilde{M}) = 2\pi(1 - \lim_{t \rightarrow \infty} f'(t))$  holds, since  $d\widetilde{M} = f dt d\theta$  and  $f'(0) = 1$ . By Theorem 1.1,

$$c(\widetilde{M}) \leq 2\pi$$

holds. Thus,  $c(\widetilde{M}) > -\infty$  means that  $\widetilde{M}$  admits a finite total curvature (if  $c(\widetilde{M})$  exists).

Let  $(M, p)$  be a complete non-compact  $n$ -dimensional Riemannian manifold with a base point  $p \in M$ . We say that  $(M, p)$  has *radial curvature at the base point  $p$  bounded from below by that of a non-compact model surface of revolution  $(\widetilde{M}, \tilde{p})$*  if, along every unit speed minimal geodesic  $\gamma : [0, a) \rightarrow M$  emanating from  $p = \gamma(0)$ , its sectional curvature  $K_M$  satisfies

$$K_M(\sigma_t) \geq G(\tilde{\gamma}(t))$$

for all  $t \in [0, a)$  and all 2-dimensional linear spaces  $\sigma_t$  spanned by  $\gamma'(t)$  and a tangent vector to  $M$  at  $\gamma(t)$ . Notice that, if the Riemannian metric of  $\widetilde{M}$  is  $dt^2 + t^2 d\theta^2$ , or  $dt^2 + \sinh^2 t d\theta^2$ , then  $G(\tilde{\gamma}(t)) = 0$ , or  $G(\tilde{\gamma}(t)) = -1$ , respectively.

For this definition, the radial curvature geometry looks artificial, **but this is not the case**, i.e., we can construct a model surface of revolution for any complete Riemannian manifold with an arbitrary given point as a base point (see [KT2, Lemma 5.1]). The existence of a  $(\widetilde{M}, \tilde{p})$  is therefore *very natural* on the above definition.

Now, we are in a point where we will state our main theorem: Let  $\mathcal{R}_M$  denote the set of all rays on  $M$  and  $\mathcal{R}_p$  the set of all rays emanating from  $p$ . Moreover, for each  $\gamma \in \mathcal{R}_M$ , let  $\Pi(\gamma)$  denote the set of all  $\alpha \in \mathcal{R}_p$  which is a limit ray of the sequence of minimal geodesic segments joining  $p$  to  $\gamma(t_i)$  for some divergent sequence  $\{t_i\}$ . Hence,  $\alpha \in \Pi(\gamma)$  is an asymptotic ray to  $\gamma$  emanating from  $p$ . Notice that  $\Pi(\gamma) = \{\gamma\}$ , if  $\gamma \in \mathcal{R}_p$ .

We set

$$A_p := \{\gamma'(0) \in \mathbb{S}_p^{n-1} \mid \gamma \in \mathcal{R}_p\},$$

where  $\mathbb{S}_p^{n-1} := \{v \in T_p M \mid \|v\| = 1\}$ , and denote by  $\text{diam}(A_p)$  the diameter of  $A_p$ . A subset  $S$  of  $A_p$  is said to be a  $\delta$ -covering of  $A_p$ , if

$$A_p \subset \bigcup_{v \in S} \overline{\mathbb{B}_\delta(v)},$$

where  $\overline{\mathbb{B}_\delta(v)} := \{w \in \mathbb{S}_p^{n-1} \mid \angle(v, w) \leq \delta\}$ .

**Main Theorem** *Let  $(M, p)$  be a complete non-compact connected Riemannian  $n$ -manifold  $M$  whose radial curvature at the base point  $p$  is bounded from below by that of a non-compact model surface of revolution  $(\widetilde{M}, \tilde{p})$ . Assume that*

(MT-1)  $c(\widetilde{M}) > \pi$ , and

(MT-2)  $\widetilde{M}$  has no pair of cut points in a sector  $\tilde{V}(\delta_0)$  for some  $\delta_0 \in (0, \pi]$ .

*Then, for any  $\gamma_1, \gamma_2, \dots, \gamma_k \in \mathcal{R}_M$  such that  $\{\alpha'(0) \in \mathbb{S}_p^{n-1} \mid \alpha \in \bigcup_{i=1}^k \Pi(\gamma_i)\}$  is a  $\delta_0$ -covering of  $A_p$ ,*

$$\max\{F_{\gamma_i} \mid i = 1, 2, \dots, k\}$$

*is an exhaustion. Moreover, if*

$$\text{diam}(A_p) \leq \delta_0,$$

*then  $F_\gamma$  is an exhaustion for all  $\gamma \in \mathcal{R}_M$ .*

The property (MT-1) does not always mean that the Gaussian curvature of  $\widetilde{M}$  is non-negative everywhere. In fact, the model surface in [KT1, Example 1.2] satisfies both properties (MT-1) and (MT-2), but  $\lim_{t \rightarrow \infty} G \circ \tilde{\gamma}(t) = -\infty$  for each meridian  $\tilde{\gamma}$ .

If a non-compact model surface of revolution  $\widetilde{M}$  admits a finite total curvature, then, for each  $\varepsilon > 0$ , there exists a compact subset  $\widetilde{K}_\varepsilon$  of  $\widetilde{M}$  such that

$$\int_{\widetilde{M} \setminus \widetilde{K}_\varepsilon} |G| d\widetilde{M} < \varepsilon.$$

Hence, we might conjecture that the Gaussian curvature of  $\widetilde{M}$  should be almost flat outside of a compact subset of  $\widetilde{M}$ . The following theorem shows that this conjecture is **false** and that the radial curvature function  $G(t)$  may change signs wildly.

**Theorem 1.3** ([TK]) *Let  $(\widetilde{M}, \tilde{p})$  be a non-compact model surface of revolution with its metric (1.1). If  $\widetilde{M}$  admits*

$$-\infty < c(\widetilde{M}) < 2\pi,$$

*then, for any  $\varepsilon > 0$ , there exists a model surface of revolution  $(\widehat{M}, \widehat{p})$  with its metric*

$$\widehat{g} = dt^2 + m(t)^2 d\theta^2, \quad (t, \theta) \in (0, \infty) \times \mathbb{S}_p^1,$$

*satisfying the differential equation  $m''(t) + \widehat{G}(t)m(t) = 0$  with initial conditions  $m(0) = 0$  and  $m'(0) = 1$ , and admitting a finite total curvature  $c(\widehat{M})$  such that*

- (1)  $\left\| G(\tilde{\gamma}(t)) - \widehat{G}(t) \right\|_{L_2} \leq \varepsilon,$
- (2)  $c(\widetilde{M}) \geq c(\widehat{M}) \geq c(\widetilde{M}) - \varepsilon$  (respectively  $c(\widetilde{M}) + \varepsilon \geq c(\widehat{M}) \geq c(\widetilde{M})$ ),
- (3)  $G(\tilde{\gamma}(t)) \geq \widehat{G}(t)$  (respectively  $\widehat{G}(t) \geq G(\tilde{\gamma}(t))$ ) on  $[0, \infty)$ , and
- (4)  $\liminf_{t \rightarrow \infty} \widehat{G}(t) = -\infty$  (respectively  $\limsup_{t \rightarrow \infty} \widehat{G}(t) = \infty$ ).

The property (MT-2) is satisfied by a von Mangoldt surface of revolution, i.e.,  $\widetilde{V}(\pi)$  has no pair of cut points. In fact, it was proved in [T] that the cut locus of a point on a von Mangoldt surface of revolution is empty or a subray of the meridian opposite to the point. The assumption (MT-2) is **not strong**. For example, consider a non-compact model surface of revolution whose radial curvature function is non-increasing (or non-positive) along a **subray** of a meridian. If the surface admits a finite total curvature, then the surface admits a sector which has no pair of cut points (see [KT2, Sector Theorem]). We do not know if (MT-2) can be removed from Main Theorem or not.

Since it is clear that  $\text{diam}(A_p) \leq \pi$ , as a corollary to Main Theorem, we get

**Theorem 1.4** ([KT1, Main Theorem]) *Let  $(M, p)$  be a complete non-compact Riemannian  $n$ -manifold  $M$  whose radial curvature at the base point  $p$  is bounded from below by that of a non-compact von Mangoldt surface of revolution  $(M^*, p^*)$ . If  $c(M^*) > \pi$ , then all Busemann functions on  $M$  are exhaustions.*

A related result for Main Theorem is Kasue's [K, Theorem 4.3], where he assumed that sectional curvature is non-negative, and he controlled diameter of each ideal boundary to be less than  $\pi/2$  in his sense.

In the following sections, all geodesics will be normalized, unless otherwise stated.

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## 2 Mass of Rays on Model Surfaces

This section is set up as a preliminary to the proof of Main Theorem (Theorem 3.6) in the next section. Throughout this section, let  $(\widetilde{M}, \tilde{p})$  denote a non-compact model surface of revolution which admits a total curvature  $c(\widetilde{M}) > \pi$ .

**Lemma 2.1** *There exists a positive number  $r_1$  such that*

$$\int_V G d\widetilde{M} > \pi + 2\Lambda_0$$

*holds for all open set  $V \subset \widetilde{M}$  containing  $B_{r_1}(\tilde{p})$  as a subset. Here we set*

$$\Lambda_0 := \frac{c(\widetilde{M}) - \pi}{3}.$$

*Proof.* Since  $c(\widetilde{M})$  is finite, for each positive number  $\varepsilon$ , there exists a positive number  $r_\varepsilon$  such that

$$\int_{\widetilde{M} \setminus B_{r_\varepsilon}(\tilde{p})} |G| d\widetilde{M} < \varepsilon$$

holds. In particular, for  $\varepsilon := \Lambda_0$ , there exists a positive number  $r_1$  such that

$$\int_{\widetilde{M} \setminus B_{r_1}(\tilde{p})} |G| d\widetilde{M} < \Lambda_0 \tag{2.1}$$

Let  $V \subset \widetilde{M}$  be an open set containing  $B_{r_1}(\tilde{p})$  as subset. It is clear that

$$\begin{aligned} \int_V G d\widetilde{M} &\geq c(\widetilde{M}) - \int_{\widetilde{M} \setminus V} |G| d\widetilde{M} \\ &\geq c(\widetilde{M}) - \int_{\widetilde{M} \setminus B_{r_1}(\tilde{p})} |G| d\widetilde{M} \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), we get

$$\int_V G d\widetilde{M} > \pi + 2\Lambda_0.$$

□

Since  $c(\widetilde{M}) > \pi$ , it follows from Cohn-Vossen's theorem [CV2, Satz 5] that  $\widetilde{M}$  has no straight line. Thus, by [SST, Lemma 6.1.1], the next lemma is clear :

**Lemma 2.2** *There exists a number  $r_2 > r_1$  such that no ray emanating from a point in  $\widetilde{M} \setminus B_{r_2}(\tilde{p})$  passes through  $B_{r_1}(\tilde{p})$ .*

**Lemma 2.3** *For each  $\tilde{q} \in \widetilde{M} \setminus B_{r_2}(\tilde{p})$ , there exists a number  $r_3 > r_2$  such that, for any  $\tilde{x} \in \widetilde{M} \setminus B_{r_3}(\tilde{p})$ ,*

$$\angle(\tilde{p}\tilde{q}\tilde{x}) \geq \frac{\pi}{2} + \Lambda_0.$$

Here  $\angle(\tilde{p}\tilde{q}\tilde{x})$  denotes the angle at the vertex  $\tilde{q}$  of the geodesic triangle  $\triangle(\tilde{p}\tilde{q}\tilde{x})$ .

*Proof.* Take any point  $\tilde{q} \in \widetilde{M} \setminus B_{r_2}(\tilde{p})$  and fix it. Let  $V_{\tilde{q}}$  denote the connected component of

$$\widetilde{M} \setminus \bigcup_{\tilde{\gamma} \in \mathcal{R}_{\tilde{q}}} \tilde{\gamma}([0, \infty))$$

containing  $B_{r_1}(\tilde{p})$ , where  $\mathcal{R}_{\tilde{q}}$  denotes the set of all rays emanating from  $\tilde{q}$ . Notice that the existence of  $V_{\tilde{q}}$  is guaranteed by Lemma 2.2, and that the boundary  $\partial V_{\tilde{q}}$  consists of two rays  $\tilde{\alpha}_+, \tilde{\alpha}_- \in \mathcal{R}_{\tilde{q}}$ , which might be the same. From Lemma 2.1,

$$c(V_{\tilde{q}}) := \int_{V_{\tilde{q}}} G d\widetilde{M} > \pi + 2\Lambda_0$$

holds. On the other hand, since  $V_{\tilde{q}}$  does not admit a ray in  $\mathcal{R}_{\tilde{q}}$ , it follows from [SST, Lemma 6.1.3] that  $c(V_{\tilde{q}})$  equals the interior angle at  $\tilde{q}$  of  $V_{\tilde{q}}$ . Hence, the interior angle at  $\tilde{q}$  of  $V_{\tilde{q}}$  is greater than  $\pi$ . Therefore, we get

$$\angle(\tilde{\alpha}'_+(0), \tilde{\alpha}'_-(0)) = 2\pi - c(V_{\tilde{q}}) < \pi - 2\Lambda_0.$$

Since  $V_{\tilde{q}}$  does not admit a ray in  $\mathcal{R}_{\tilde{q}}$  and  $\tilde{\alpha}_+, \tilde{\alpha}_-$  are symmetric under the reflection with respect to the meridian  $\mu_{\tilde{q}}$  passing through  $\tilde{q}$ ,

$$\begin{aligned} \max\{\angle(\tilde{\gamma}'(0), \mu'_{\tilde{q}}(d(\tilde{p}, \tilde{q}))) \mid \tilde{\gamma} \in \mathcal{R}_{\tilde{q}}\} &= \angle(\tilde{\alpha}'_+(0), \mu'_{\tilde{q}}(d(\tilde{p}, \tilde{q}))) \\ &= \angle(\tilde{\alpha}'_-(0), \mu'_{\tilde{q}}(d(\tilde{p}, \tilde{q}))) \\ &< \frac{\pi}{2} - \Lambda_0. \end{aligned} \tag{2.3}$$

In particular, by (2.3),

$$\angle(\tilde{\gamma}'(0), \mu'_{\tilde{q}}(d(\tilde{p}, \tilde{q}))) < \frac{\pi}{2} - \Lambda_0$$

holds for all  $\tilde{\gamma} \in \mathcal{R}_{\tilde{q}}$ .

Let  $\tilde{\alpha} : [0, d(\tilde{q}, \tilde{x})] \rightarrow \widetilde{M}$  denote a minimal geodesic segment joining  $\tilde{q}$  to a point  $\tilde{x} \in \widetilde{M}$ . If  $d(\tilde{q}, \tilde{x})$  is sufficient large, then  $\tilde{\alpha}'(0)$  is close to some  $\tilde{\gamma}'(0)$ ,  $\tilde{\gamma} \in \mathcal{R}_{\tilde{q}}$ . Therefore, there exists a number  $r_3 > r_2$  such that, for any minimal geodesic segment  $\tilde{\alpha} : [0, d(\tilde{q}, \tilde{x})] \rightarrow \widetilde{M}$  joining  $\tilde{q}$  to  $\tilde{x}$  with  $d(\tilde{q}, \tilde{x}) > r_3$ ,

$$\angle(\tilde{\alpha}'(0), \mu'_{\tilde{q}}(d(\tilde{p}, \tilde{q}))) < \frac{\pi}{2} - \Lambda_0. \tag{2.4}$$

The equation (2.4) implies that

$$\angle(\tilde{p}\tilde{q}\tilde{x}) \geq \frac{\pi}{2} + \Lambda_0$$

for all  $\tilde{x} \in \widetilde{M} \setminus B_{r_3}(\tilde{p})$ . □

### 3 Proof of Main Theorem

Our purpose of this section is to prove Main Theorem (Theorem 3.6). In the proof of the theorem, we will apply a new type of the Toponogov comparison theorem. The comparison theorem was established by the present authors as generalization of the comparison theorem in conventional comparison geometry, which is stated as follows:

#### A New Type of Toponogov Comparison Theorem ([KT2, Theorem 4.12])

*Let  $(M, p)$  be a complete non-compact Riemannian manifold  $M$  whose radial curvature at the base point  $p$  is bounded from below by that of a non-compact model surface of revolution  $(\widetilde{M}, \tilde{p})$ . If  $(\widetilde{M}, \tilde{p})$  admits a sector  $\widetilde{V}(\delta_0)$ ,  $\delta_0 \in (0, \pi]$ , having no pair of cut points, then, for every geodesic triangle  $\triangle(pxy)$  in  $(M, p)$  with  $\angle(xpy) < \delta_0$ , there exists a geodesic triangle  $\widetilde{\triangle}(p\tilde{x}\tilde{y}) := \triangle(\tilde{p}\tilde{x}\tilde{y})$  in  $\widetilde{V}(\delta_0)$  such that*

$$d(\tilde{p}, \tilde{x}) = d(p, x), \quad d(\tilde{p}, \tilde{y}) = d(p, y), \quad d(\tilde{x}, \tilde{y}) = d(x, y) \quad (3.1)$$

and that

$$\angle(xpy) \geq \angle(\tilde{x}\tilde{p}\tilde{y}), \quad \angle(pxy) \geq \angle(\tilde{p}\tilde{x}\tilde{y}), \quad \angle(pyx) \geq \angle(\tilde{p}\tilde{y}\tilde{x}). \quad (3.2)$$

Here  $\angle(xpy)$  denotes the angle between the minimal geodesic segments from  $x$  to  $p$  and  $y$  forming the triangle  $\triangle(pxy)$ .

**Remark 3.1** In [KT3], the present authors very recently generalized, from the radial curvature geometry's standpoint, the Toponogov comparison theorem to a complete Riemannian manifold with smooth convex boundary.

Hereafter, let  $(M, p)$  denote a complete non-compact Riemannian  $n$ -manifold  $M$  whose radial curvature at the base point  $p$  is bounded from below by that of a non-compact model surface of revolution  $(\widetilde{M}, \tilde{p})$  with its metric (1.1),  $\mathcal{R}_M$  the set of all rays on  $M$ , and  $\mathcal{R}_p$  the set of all rays emanating from  $p$ . Moreover, for each  $\gamma \in \mathcal{R}_M$ , let  $\Pi(\gamma)$  denote the set of all  $\alpha \in \mathcal{R}_p$  which is a limit ray of the sequence of minimal geodesic segments joining  $p$  to  $\gamma(t_i)$  for some divergent sequence  $\{t_i\}$ . Furthermore, we assume that

(MTI-1)  $c(\widetilde{M}) > \pi$ , and

(MTI-2)  $\widetilde{M}$  has no pair of cut points in a sector  $\widetilde{V}(\delta_0)$  for some  $\delta_0 \in (0, \pi]$ .



**Lemma 3.2** *Let  $\gamma \in \mathcal{R}_M$  and  $\alpha : [0, d(p, q)] \rightarrow M$  a minimal geodesic segment joining  $p$  to a point  $q \in M \setminus B_{r_2}(p)$  such that*

$$\angle(\alpha'(0), \beta'_\gamma(0)) < \delta_0$$

*for some  $\beta_\gamma \in \Pi(\gamma)$ . Then,*

$$\angle(\sigma'(0), \alpha'(d(p, q))) \leq \frac{\pi}{2} - \Lambda_0$$

*holds for a ray  $\sigma$  emanating from  $q$  asymptotic to  $\gamma$ . Here  $\Lambda_0$  and  $r_2$  denote the positive numbers guaranteed in Lemmas 2.1 and 2.2, respectively.*

*Proof.* Since  $\beta_\gamma \in \Pi(\gamma)$ , there exists a divergent sequence  $\{t_i\}$  such that the sequence of minimal geodesic segments  $\beta_i : [0, d(p, \gamma(t_i))] \rightarrow M$  joining  $p$  to  $\gamma(t_i)$  convergent to  $\beta_\gamma$ . Since  $\lim_{t \rightarrow 0} \angle(\beta'_i(0), \beta'_\gamma(0)) = 0$ , there is a number  $i_0 \in \mathbb{N}$  such that

$$\angle(\beta'_i(0), \alpha'(0)) < \delta_0$$

for all  $i \geq i_0$ . Thus, by the new type of the Toponogov comparison theorem, there exists a geodesic triangle  $\tilde{\Delta}(p\gamma(t_i)q) \subset \tilde{V}(\delta_0)$  corresponding to the triangle  $\Delta(p\gamma(t_i)q)$ ,  $i \geq i_0$ , such that (3.1) holds for  $x = \gamma(t_i)$  and  $y = q$ , and that

$$\angle(-\alpha'(d(p, q)), \sigma'_i(0)) \geq \angle(\tilde{p}\tilde{q}\tilde{\gamma}(t_i)).$$

Here  $\sigma_i : [0, d(q, \gamma(t_i))] \rightarrow M$  denotes a minimal geodesic segment joining  $q$  to  $\gamma(t_i)$ . By Lemma 2.3, we get

$$\angle(-\alpha'(d(p, q)), \sigma'_i(0)) \geq \frac{\pi}{2} + \Lambda_0$$

for sufficiently large  $i$ . Hence,

$$\angle(-\alpha'(d(p, q)), \sigma'(0)) \geq \frac{\pi}{2} + \Lambda_0$$

where  $\sigma$  denotes a limit ray of the sequence  $\{\sigma_i\}$ , which is asymptotic to  $\gamma$ .  $\square$

Hereafter, let  $F_\gamma$  denote a Busemann function of a  $\gamma \in \mathcal{R}_M$ . Notice that, by the definition of  $F_\gamma$ ,  $|F_\gamma(x) - F_\gamma(y)| \leq d(x, y)$  holds for all  $x, y \in M$ , i.e.,  $F_\gamma$  is Lipschitz continuous with Lipschitz constant 1. Hence,  $F_\gamma$  is differentiable except for a measure zero set. Moreover, we have

**Proposition 3.3** ([KT1, Theorem 3.1]) *Let  $\gamma$  be a ray on a complete non-compact Riemannian manifold  $M$ . Then,  $F_\gamma$  is differentiable at a point  $q \in M$  if and only if there exists a unique ray emanating from  $q$  asymptotic to  $\gamma$ . Moreover, the gradient vector of  $F_\gamma$  at a differentiable point  $q$  equals the velocity vector of the unique ray asymptotic to  $\gamma$ .*

**Lemma 3.4** *Let  $\gamma \in \mathcal{R}_M$  and  $\alpha : [0, d(p, q)] \rightarrow M$  a minimal geodesic segment joining  $p$  to a point  $q \in M \setminus B_{r_2}(p)$  such that*

$$\angle(\alpha'(0), \beta'_\gamma(0)) < \delta_0$$

*for some  $\beta_\gamma \in \Pi(\gamma)$ . If  $F_\gamma$  is differentiable at  $\alpha(t)$  for almost all  $t \in (a, b) \subset (r_2, d(p, q)]$ , then*

$$F_\gamma(\alpha(b)) - F_\gamma(\alpha(a)) \geq (b - a) \sin \Lambda_0.$$

*Proof.* Assume that  $F_\gamma$  is differentiable at  $\alpha(t_0)$ ,  $t_0 \in (a, b)$ . By Lemma 3.2 and Proposition 3.3, we get

$$\angle((\nabla F_\gamma)_{\alpha(t_0)}, \alpha'(t_0)) \leq \frac{\pi}{2} - \Lambda_0$$

Hence, for almost all  $t \in (a, b)$ ,

$$\frac{d}{dt}F_\gamma(\alpha(t)) = \langle (\nabla F_\gamma)_{\alpha(t)}, \alpha'(t) \rangle = \cos(\angle((\nabla F_\gamma)_{\alpha(t)}, \alpha'(t))) \geq \sin \Lambda_0$$

It follows from Dini's theorem [D] (cf. [H, Section 2.3], [WZ, Theorem 7.29]) that

$$F_\gamma(\alpha(b)) - F_\gamma(\alpha(a)) = \int_a^b \frac{d}{dt}F_\gamma(\alpha(t)) dt \geq (b - a) \sin \Lambda_0.$$

□

**Lemma 3.5** *Let  $\gamma \in \mathcal{R}_M$  and  $\alpha : [0, d(p, q)] \rightarrow M$  a minimal geodesic segment joining  $p$  to a point  $q \in M \setminus B_{r_2}(p)$  such that*

$$\angle(\alpha'(0), \beta'_\gamma(0)) \leq \delta_0$$

*for some  $\beta_\gamma \in \Pi(\gamma)$ . Then,*

$$F_\gamma(q) - F_\gamma(\alpha(r_2)) \geq (d(p, q) - r_2) \sin \Lambda_0 \quad (3.3)$$

*holds.*

*Proof.* First, we will prove (3.3) under the assumption that

$$\angle(\alpha'(0), \beta'_\gamma(0)) < \delta_0.$$

The general case will be completed by the limit argument. If we prove that, for each  $t_0 \in (r_2, d(p, q))$ , there exists a number  $\varepsilon_0 > 0$  such that

$$F_\gamma(\alpha(t)) - F_\gamma(\alpha(s)) \geq (t - s) \sin \Lambda_0 \quad (3.4)$$

holds for all  $s, t \in (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$  with  $s < t$ , then the equation (3.3) is clear.

Take any  $t_0 \in (r_1, d(p, q))$ , and fix it. Since  $\alpha$  is minimal on  $[0, d(p, q)]$ ,  $\alpha(t_0)$  is not a cut point of  $p = \alpha(0)$ . Hence, there exist an open neighborhood  $\mathcal{U} \subset \mathbb{S}_p^{n-1}$  around  $\alpha'(0)$ , an open neighborhood  $U$  around  $\alpha(t_0)$ , and an open interval  $(t_0 - \varepsilon_0, t_0 + \varepsilon_0)$  such that  $\mathcal{U} \times (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$  is diffeomorphic to  $U$  by a map  $\varphi$ , where  $\varphi^{-1}(v, t) := \exp_p(tv)$ . Since  $F_\gamma \circ \varphi^{-1}$  is Lipschitz, it follows from Rademacher's theorem (cf. [Mo]) that there exists a set  $\mathcal{E} \subset T_p M$  of Lebesgue measure zero such that  $F_\gamma \circ \varphi^{-1}$  is differentiable on  $(\mathcal{U} \times (t_0 - \varepsilon_0, t_0 + \varepsilon_0)) \setminus \mathcal{E}$ . Moreover, for each  $v \in \mathcal{U}$ , we set

$$\mathcal{E}_v := \{t \in (t_0 - \varepsilon_0, t_0 + \varepsilon_0) \mid (v, t) \in \mathcal{E}\}.$$

Remark that the set  $\mathcal{E}_v$  has also Lebesgue measure zero for all most all  $v \in \mathcal{U}$  (cf. [WZ, Lemma 6.5]). Thus, we may find a sequence  $\{\alpha_j\}$  of minimal geodesic segments emanating

from  $p$  converging to  $\alpha$  such that each  $F_\gamma$  is differentiable at  $\alpha_j(t)$  for almost all  $t \in (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$ . By Lemmas 3.2 and 3.4, for each  $j \in \mathbb{N}$ ,

$$F_\gamma(\alpha_j(t)) - F_\gamma(\alpha_j(s)) \geq (t - s) \sin \Lambda_0$$

holds for all  $s, t \in (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$  with  $s < t$ . Then, by taking the limit, we get (3.4).

Assume that

$$\angle(\alpha'(0), \beta'_\gamma(0)) = \delta_0.$$

It is clear that there exists a sequence  $\{\alpha_i : [0, \ell_i] \rightarrow M\}$  of minimal geodesic segments  $\alpha_i$  emanating from  $p = \alpha_i(0)$  convergent to  $\alpha$  such that  $\angle(\alpha'_i(0), \beta'_\gamma(0)) < \delta_0$  for each  $i \in \mathbb{N}$ . From the argument above,

$$F_\gamma(\alpha_i(\ell_i)) - F_\gamma(\alpha_i(r_2)) \geq (\ell_i - r_2) \sin \Lambda_0$$

By taking the limit, we get (3.3). □

Set

$$A_p := \{\gamma'(0) \in \mathbb{S}_p^{n-1} \mid \gamma \in \mathcal{R}_p\},$$

and denote by  $\text{diam}(A_p)$  the diameter of  $A_p$ . Then, we have our main theorem in this article :

**Theorem 3.6** *For any  $\gamma_1, \gamma_2, \dots, \gamma_k \in \mathcal{R}_M$  such that  $\{\alpha'(0) \in \mathbb{S}_p^{n-1} \mid \alpha \in \bigcup_{i=1}^k \Pi(\gamma_i)\}$  is a  $\delta_0$ -covering of  $A_p$ ,*

$$\max\{F_{\gamma_i} \mid i = 1, 2, \dots, k\}$$

*is an exhaustion. Moreover, if  $\text{diam}(A_p) \leq \delta_0$ , or  $\delta_0 = \pi$ , then  $F_\gamma$  is an exhaustion for all  $\gamma \in \mathcal{R}_M$ .*

*Proof.* Suppose that  $\max\{F_{\gamma_i} \mid i = 1, 2, \dots, k\}$  is not an exhaustion, i.e., for some  $a \in \mathbb{R}$ ,

$$X := \bigcap_{i=1}^k F_{\gamma_i}^{-1}(-\infty, a]$$

is non-compact. Hence, there exists a sequence  $\{q_j\}$  of points  $q_j \in X$  such that

$$\lim_{j \rightarrow \infty} d(p, q_j) = \infty.$$

Let  $\alpha_j : [0, d(p, q_j)] \rightarrow M$  denote a minimal geodesic segment joining  $p$  to  $q_j$ . Since  $\lim_{j \rightarrow \infty} d(p, q_j) = \infty$ , there exists a number  $j_0 \in \mathbb{N}$  such that

$$r_2 < d(p, q_j)$$

for all  $j \geq j_0$ . Furthermore, by choosing an infinite subsequence of  $\{\alpha_j\}$ , we may assume that there exist  $i_0 \in \{1, 2, \dots, k\}$  such that, for each  $j \geq j_0$ ,

$$\angle(\alpha'_j(0), \beta'_{\gamma_{i_0}}(0)) \leq \delta_0$$

holds for some  $\beta_{\gamma_j} \in \Pi(\gamma_{i_0})$ . It follows from Lemma 3.5 that

$$F_{\gamma_{i_0}}(q_j) - F_{\gamma_{i_0}}(\alpha_j(r_2)) \geq (d(p, q_j) - r_2) \sin \Lambda_0$$

for all  $j \geq j_0$ . Since  $q_j \in F_{\gamma_{i_0}}^{-1}(-\infty, a]$  for all  $j \geq j_0$ ,

$$a - F_{\gamma_{i_0}}(\alpha_j(r_2)) \geq (d(p, q_j) - r_2) \sin \Lambda_0.$$

Since  $\lim_{j \rightarrow \infty} d(p, q_j) = \infty$ , we have  $\lim_{j \rightarrow \infty} F_{\gamma_{i_0}}(\alpha_j(r_2)) = -\infty$ . This is impossible, since  $|F_{\gamma_{i_0}}(p) - F_{\gamma_{i_0}}(\alpha_j(r_2))| \leq d(p, \alpha_j(r_2)) = r_2$  for all  $j \geq j_0$ . Therefore,  $\max\{F_{\gamma_i} \mid i = 1, 2, \dots, k\}$  is an exhaustion.

Next, we will prove the second claim. Assume that  $\text{diam}(A_p) \leq \delta_0$ . Since  $\angle(v, w) \leq \delta_0$  for all  $v, w \in A_p$ , it is clear that  $\{v\}$  is a  $\delta_0$ -covering of  $A_p$  for each  $v \in A_p$ . Hence, for each  $\gamma \in \mathcal{R}_M$ ,  $\{\alpha'(0) \in \mathbb{S}_p^{n-1} \mid \alpha \in \Pi(\gamma)\}$  is a  $\delta_0$ -covering of  $A_p$ . From the argument above, this implies that  $F_\gamma$  is an exhaustion for all  $\gamma \in \mathcal{R}_M$ . If  $\delta_0 = \pi$ , then the claim is clear, since  $\text{diam}(A_p) \leq \pi$ .  $\square$

From the same argument in [KT1, Section 4.2], we get

**Corollary 3.7** *The isometry group  $I(M)$  of  $M$  is compact, if  $\text{diam}(A_p) \leq \delta_0$ , or  $\delta_0 = \pi$ .*

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